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AN ANALYTICAL SOLUTION FOR CROSS-PLY LAMINATES UNDER CYLINDRICAL BENDING BASED ON THROUGH-THE-THICKNESS INEXTENSIBILITY. PART I—STATIC LOADING

S. JAVAD JALALI and FARID TAHERI
Department of Civil Engineering, Technical University of Nova Scotia, P.O. Box 1000, Halifax, Nova Scotia, Canada B3J 2X4

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Abstract—The response of cross-ply laminated plates under cylindrical and/or planar bending is examined. A novel method, based on the assumption of inextensibility of the plate through the thickness, is introduced. The solution for a plate with simple supports at two ends is obtained. The numerical results for several cases are presented. Displacement and stresses obtained by the introduced method agree very well with the exact elasticity solution. The presented method, in addition to its simplicity in comparison with the exact solution, has the capability of being used for dynamic problems. © 1998 Elsevier Science Ltd.

INTRODUCTION

Classical laminated plate theory based on Kirchhoff hypothesis was presented and discussed by Reissner and Stavsky (1961), Dong *et al.* (1961), and Ashton and Whitney (1970). The inadequacy of this theory in determining the stresses and deflections of thick laminates has led to the development of several approximate methods and some exact solutions. The exact solutions for some limited cases under static loading were developed by Pagano (1969, 1970a, 1970b). In spite of the limitations of these solutions they have been very useful for academic and research purposes.

The exact solutions of laminated plates for dynamic cases were given by Srinivas *et al.* (1970), Jones (1970, 1971) and Kulkari and Pagano (1972). Jones (1970) presented the solution for cross-ply laminates. His solution encountered problem when the number of layers became moderately large and, therefore, his work was limited to two-layer plates. Kulkari and Pagano (1972) had the same problem for angle-ply laminates and their work was limited to five-layer plates. They also mentioned that the solution for large mh/L ratios led to inaccurate results, where m, h and L are mode number, thickness and length of plate, respectively. Srinivas *et al.* (1970) presented the solution for cross-ply laminated plates with simple supports at all four edges. They applied the solution to three-layer plates. Because of the severe limitations, the exact solutions have not been used widely. However, for more general cases, the use of some approximate theory has been necessary.

Among the approximate methods are the so-called high-order laminated plate theories, in which the simplest and the most popular is the first-order theory. This theory, which is similar to Mindlin's work (1951) for isotropic plates, was expanded for the laminated plates by Yang *et al.* (1966), Whitney and Pagano (1970) and Chow (1971). In spite of very good results of this theory in predicting the deflections, the theory does not improve the accuracy of stresses over those obtained by classical plate theory. Whitney (1972) used the results of the first-order theory to refine the stresses. Rehfield and Valisetty (1983) developed a similar method, but they used the results of classical plate theory. Many different higher-order theories have been proposed which are intended to improve the accuracy of stresses through the thickness of the plate. Some of these theories were presented by Whitney and Sun (1973), Lo *et al.* (1977), and Reddy (1984, 1989).

Despite the success in predicting the stresses under distributed static loading by using the proposed higher-order theories, the results are not satisfactory for concentrated load and dynamic problems. The inaccuracy of the results increases for higher modes of vibration. This problem can lead to quite inaccurate results when the behavior of plates subjected to concentrated impact is considered. Goldsmith (1960) showed that the influence of modes with high natural frequencies for impact loading is significant. Furthermore, as Jalali and Taheri (1998) will show, the high-order theories are not capable in accurate predicting the stresses under these modes. Recognizing these two anomalies, one can expect a high inaccuracy in the result of stresses obtained by one of the high-order theories. To overcome this problem, it is necessary to include more higher-order terms in the power series expansions of the assumed in-plane displacement field. The other shortfall concerning the concentrated impact, is due to the local effect of the concentrated load which leads to a rapid variation of stresses in the vicinity of the loading region. To resolve this problem one needs to include higher-order terms in the assumed out-of-plane displacement field (similar to the work done by Lo *et al.* (1977)). However, the complexity of the formulation arising from the inclusion of each new higher-order term makes the procedure impractical.

Due to the inherent shortfalls of the above-mentioned higher-order theories in treating laminated plates, one has to resort to elasticity based finite element analysis. For finite element analysis, one must discretize the thickness of the plates by several elements. To ensure a reasonable aspect ratio for the elements, one has to adopt large number of elements on the width and length directions. Moreover, to accommodate localized effect, such as the effect of concentrated loads and cut-outs, one must further refine the mesh. Given the fact that elements representing beams and plates have 2 and 3 degrees-of-freedom per node, an ordinary laminated plate or beam may require several thousands of degree-of-freedoms for adequate simulation of its behavior. Consequently, one often encounters difficulties when treating such problems by the ordinary computational facilities.

The trend to reach to a more practical and accurate method has led to the development of various layerwise laminate theories proposed by several researcher [see for example Robbins and Reddy (1993); Basar *et al.* (1993); Reddy (1989)]. In these theories the thickness of the laminate is divided to several layers in which the variation of displacement components through the thickness are assumed to vary linearly or are represented by a one-dimensional (1-D) Lagrangian polynomials. The accuracy of the results produced by these theories are comparable to elasticity based finite element analysis. Although the total number of degrees-of-freedom in layerwise theories are comparable to elasticity based finite element analysis, the volume of the input data is considerably reduced. Moreover, the refinement of mesh through the thickness of the laminate can be done independently of those on the in-plane directions. Neglecting the through-the-thickness strain produces a model with about two-thirds the degrees-of-freedom of a comparable finite element model. Nonetheless, the areas with localized effect cannot be modeled accurately.

The purpose of the present work is to develop an analytical semi-exact method which is applicable to dynamic and impact problems. The basis of this method is the assumption of the through-the-thickness inextensibility of laminates. A solution for cross-ply laminated plates under cylindrical and planar bending is obtained. In addition to its accuracy, the proposed method in comparison to the exact solution, is simple and more practical. Also, the solution is easily extendible for dynamic problems.

It is worthwhile to mention that as for most analytical solutions, the applicability of the proposed solution is limited to structures with simple geometry. Nevertheless, the solution is useful for preliminary investigations and research purposes. It can also be used as a mean for validating the results obtained by other approximate, yet practical methods.

FORMULATION

We use the engineering strain, γ_{xz} and γ_{yz}, for developing a displacement field through the thickness of plate. These strains are defined as:

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{1}$$

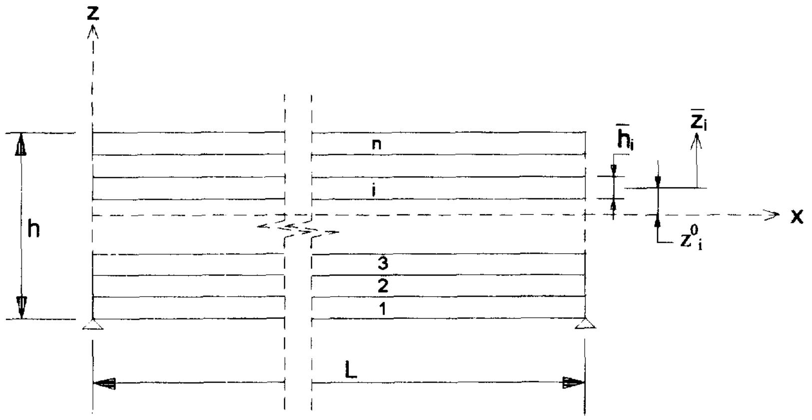

Fig. 1. The coordinate systems of laminates.

where u and v are the in-plane displacements in the x- and y-directions, respectively, and w denotes the out-of-plane displacement in the z-direction. If we assume w is constant through the thickness, the integration of eqn (1) with respect to z leads to a displacement field in the x- and y-directions as follows:

$$u = -z\frac{\partial w}{\partial x} + u^0 + \int_0^z \gamma_{xz}\,\mathrm{d}z \tag{2a}$$

$$v = -z\frac{\partial w}{\partial y} + v^0 + \int_0^z \gamma_{yz}\,\mathrm{d}z \tag{2b}$$

where u^0 and v^0 are the displacements at $z = 0$. The assumption of a constant γ_{xz} and γ_{yz} through the thickness in the above equations leads to the solution of Mindlin (1951) for isotropic plates and of Yang *et al.* (1966), Whitney and Pagano (1970) and Chow (1971) for laminated plates. The assumption of linear variation of γ_{xz} and γ_{yz} leads to the solution of Whitney and Sun (1973) and by taking a parabolic function for γ, one obtains the solution of Reissner (1945, 1975) and Reddy (1984) for isotropic and laminated plates, respectively. However, in this paper we are looking for the exact form of γ. For this purpose we consider a laminate composed of n orthotropic layers, such that the various axes of material symmetry are parallel to the plate axes x, y and z.

As shown in Fig. (1), the global Cartesian axes are constructed on the center line of the laminate at $x = 0$. Also, a local coordinate is defined on the center line of each layer. Since we are only concerned with the variables in the x-direction, for simplicity, we omit the subscripts. Therefore, σ and ε are the stress and strain in the x-direction and τ is the through-the-thickness shear stress in the xz plane, respectively. The laminate is simply-supported at two ends. Therefore, the boundary conditions at two ends are defined as:

$$x = 0, L \begin{cases} w = 0 \\ \sigma_i = 0 \quad i = 1, 2, \ldots, n \end{cases} \tag{3}$$

where subscript i indicates the layer number and n is the total number of layers. The constitutive relationship within each layer for the cylindrical and planar bending can be simplified in the following identical form:

$$\begin{aligned} \sigma &= \bar{E}\varepsilon \\ \tau &= G\gamma, \end{aligned} \tag{4}$$

where G and $\bar{E}$ are the corresponding shear modulus in zy plane and the longitudinal stiffness in x-direction, respectively. In the case of the beam type problems (planar bending) the longitudinal stiffness is equal to the modulus of elasticity in the x-direction. However, for cylindrical bending it is equal to:

$$\bar{E} = \frac{E_{11}}{1-\upsilon_{12}\upsilon_{21}}, \tag{5}$$

where E and υ are the elastic modulus and the Poisson's ratio of the layer and subscripts 1 and 2 refer to the x- and y-direction, respectively.

The first derivative of eqn (2a) with respect to x gives the longitudinal strain, ε. Multiplying the result by the longitudinal stiffness, we can find the longitudinal stress of each layer as:

$$\sigma_i = -\bar{z}_i\bar{E}_i\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \bar{E}_i\frac{\mathrm{d}u_i^0}{\mathrm{d}x} + \frac{\bar{E}_i}{G_i}\int_0^{\bar{z}_i}\frac{\mathrm{d}\tau_i}{\mathrm{d}x}\mathrm{d}\bar{z}_i. \tag{6}$$

Recognizing that there is no body force in the x-direction, the equilibrium condition in the x-direction is:

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}x} + \frac{\mathrm{d}\tau_i}{\mathrm{d}\bar{z}} = 0. \tag{7}$$

Substituting eqn (6) in eqn (7) gives:

$$\frac{\mathrm{d}\tau_i}{\mathrm{d}\bar{z}_i} = \bar{z}_i\bar{E}_i\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} - \bar{E}_i\frac{\mathrm{d}^2 u_i^0}{\mathrm{d}x^2} - \frac{\bar{E}_i}{G_i}\int_0^{\bar{z}_i}\frac{\mathrm{d}^2\tau_i}{\mathrm{d}x^2}\mathrm{d}\bar{z}_i. \tag{8}$$

Since almost any load can be expressed in the form of Fourier series, we assume:

$$q(x) = q_0 \sin px, \tag{9}$$

where q_0 is constant and

$$p = \frac{m\pi}{L}.$$

The solution of eqn (8) is of the form

$$w = w_0 \sin px \tag{10a}$$

$$\tau_i = B_i \cos px \tag{10b}$$

$$u_i^0 = c_i \cos px, \tag{10c}$$

where B_i is a function of $\bar{z}_i$, but w_0 and c_i are constants. Substituting eqn (10) in eqn (8) leads to

$$\frac{\mathrm{d}B_i}{\mathrm{d}\bar{z}_i} = -\bar{z}_i\bar{E}_i w_0 p^3 + c_i\bar{E}_i p^2 + \frac{\bar{E}_i}{G_i}p^2\int_0^{\bar{z}_i} B_i\,\mathrm{d}\bar{z}_i. \tag{11}$$

Equations (10) satisfy the boundary conditions of simple supports at the two ends defined by eqn (3). Recognizing eqn (10b), the boundary conditions on the bottom and top of each layer are defined as:

$$\tau_i^{\mathrm{B}} = \tau_i^- \cos px$$
$$\tau_i^{\mathrm{T}} = \tau_i^+ \cos px, \tag{12a}$$

or

$$B_i(-\bar{h}_i/2) = \tau_i^-$$
$$B_i(+\bar{h}_i/2) = \tau_i^+. \tag{12b}$$

Using the Laplace transformation and applying the boundary conditions at $\bar{z}_i = \pm\bar{h}_i/2$, we can find the solution of the above differential equation. That is:

$$B_i = \left(\frac{\tau_i^+ + \tau_i^-}{2}\right)\frac{\cosh(\sqrt{\beta_i}\bar{z}_i)}{\cosh(\sqrt{\beta_i}\bar{h}_i/2)} + \left(\frac{\tau_i^+ - \tau_i^-}{2}\right)\frac{\sinh(\sqrt{\beta_i}\bar{z}_i)}{\sinh(\sqrt{\beta_i}\bar{h}_i/2)} + w_0 G_i p\left[1 - \frac{\cosh(\sqrt{\beta_i}\bar{z}_i)}{\cosh(\sqrt{\beta_i}\bar{h}_i/2)}\right] \tag{13a}$$

$$c_i = \frac{\tau_i^+ - \tau_i^-}{2\bar{E}_i}\frac{\sqrt{\beta_i}}{\sinh(\sqrt{\beta_i}\bar{h}_i/2)} \tag{13b}$$

where

$$\beta_i = \frac{\bar{E}_i p^2}{G_i}. \tag{14}$$

Substituting eqns (13) and (10) in eqn (6) gives the relation for longitudinal stress:

$$\sigma_i = p\bar{E}_i\left[\begin{array}{l}\dfrac{w_0 p}{\sqrt{\beta_i}}\dfrac{\sinh(\sqrt{\beta_i}\bar{z}_i)}{\cosh(\sqrt{\beta_i}\bar{h}_i/2)} - \left(\dfrac{\tau_i^+ + \tau_i^-}{2G_i\sqrt{\beta_i}}\right)\dfrac{\sinh(\sqrt{\beta_i}\bar{z}_i)}{\cosh(\sqrt{\beta_i}\bar{h}_i/2)} \\ -\left(\dfrac{\tau_i^+ - \tau_i^-}{2G_i\sqrt{\beta_i}}\right)\dfrac{\cosh(\sqrt{\beta_i}\bar{z}_i)}{\sinh(\sqrt{\beta_i}\bar{h}_i/2)}\end{array}\right]\sin px. \tag{15}$$

Integrating eqn (13a) over the thickness and substituting in eqn (10b) gives the shear force of each layer. Since the shear force at any section of the laminate is equal to the sum of the shear forces carried by layers and recognizing that:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q, \tag{16}$$

where V is the shear force of the section, leads to

$$\sum_{i=1}^{n}\frac{\tau_i^+ + \tau_i^-}{\sqrt{\beta_i}}\tanh(\sqrt{\beta_i}\bar{h}_i/2) + w_0 p\sum_{i=1}^{n}G_i\left[\bar{h}_i - \frac{2}{\sqrt{\beta_i}}\tanh(\sqrt{\beta_i}\bar{h}_i/2)\right] = \frac{q_0}{p}. \tag{17}$$

Substituting eqns (13) and (10) in eqn (2a) gives the longitudinal displacement of each layer:

$$u_i = \begin{bmatrix} \dfrac{\tau_i^+ + \tau_i^-}{2G_i\sqrt{\beta_i}} \dfrac{\sinh(\sqrt{\beta_i}\bar{z}_i)}{\cosh(\sqrt{\beta_i}\bar{h}_i/2)} + \dfrac{\tau_i^+ - \tau_i^-}{2G_i\sqrt{\beta_i}} \dfrac{\cosh(\sqrt{\beta_i}\bar{z}_i)}{\sinh(\sqrt{\beta_i}\bar{h}_i/2)} \\ -\dfrac{w_0 p}{\sqrt{\beta_i}} \dfrac{\sinh(\sqrt{\beta_i}\bar{z}_i)}{\cosh(\sqrt{\beta_i}\bar{h}_i/2)} \end{bmatrix} \cos px. \tag{18}$$

For a laminate consisting of n layers there are n unknowns, $n-1$ shear stresses at the interface of adjacent layers and one w_0. Equating the longitudinal displacements, obtained by eqn (18), of each of the two adjacent layers at interfaces, gives $n-1$ equations. These equations in addition to eqn (17), provide an adequate number of equations to solve for n unknowns. When the unknowns are obtained, the distribution of stresses can be found by eqns (13a) and (15).

For a laminate consisting of only one unidirectional layer the solution is simplified as follows:

$$w_0 = \frac{q_0}{p^2 Gh} \frac{1}{1 - \dfrac{\tanh \alpha}{\alpha}} \tag{19a}$$

$$\sigma = \frac{1}{2} \times \frac{q_0 \phi^2}{\alpha - \tanh(\alpha)} \times \frac{\sinh(2\alpha z/h)}{\cosh(\alpha)} \sin px \tag{19b}$$

$$\tau = \frac{1}{2} \times \frac{q_0 \phi}{\alpha - \tanh(\alpha)} \left[1 - \frac{\cosh(2\alpha z/h)}{\cosh(\alpha)}\right] \cos px, \tag{19c}$$

where

$$\alpha = \sqrt{\beta} h/2$$
$$\phi = \sqrt{\bar{E}/G}. \tag{20}$$

NUMERICAL RESULTS

In this part we verify the integrity of the introduced method. The deflections and the stresses are obtained for several laminates and the results are compared with the exact elasticity solution (Pagano, 1969) and those of classical laminated plate theory (CLPT). Four configurations of layup under two loading distributions, a half a sine and a concentrated load at midspan, are considered. The four configurations are as follows:

Case 1—unidirectional laminate with the fibers oriented in the x-direction (0° laminate).

Case 2—an asymmetric laminate composed of one 0° and one 90° layer with equal thickness [0/90].

Case 3—laminate composed of three layers with equal thickness [0/90/0].

Case 4—laminate composed of five layers with equal thickness [0/90/0/90/0]. In all cases, we assume the layers have the following properties

$$E_L = 174.6 \text{ GPa} \quad E_r = 7 \text{ GPa}$$
$$G_{LT} = 3.5 \text{ GPa} \quad G_{TT} = 1.4 \text{ GPa}$$
$$v_{LT} = v_{TT} = 0.25, \tag{21}$$

where L and T are the directions parallel and normal to the fibers, respectively.

First the behavior of the laminates subjected to a half sine distributed load is considered. Deflections for plates with different length to depth ratios, L/h, by three different methods, were obtained. The results were normalized against those of CLPT and the exact

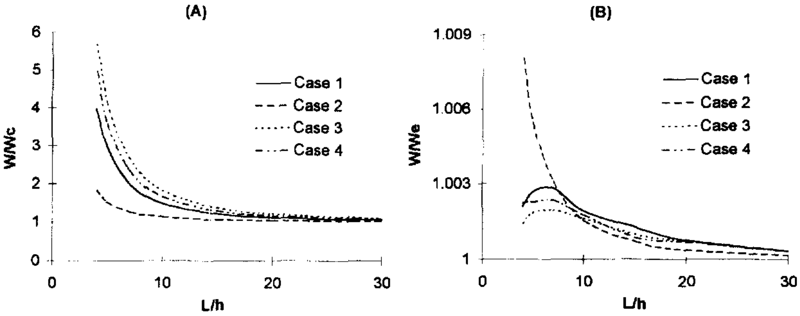

Fig. 2. Comparison of the midspan deflections for the various cases: (a) values normalized with respect to that of CLPT; (b) values normalized with respect to that of exact solution.

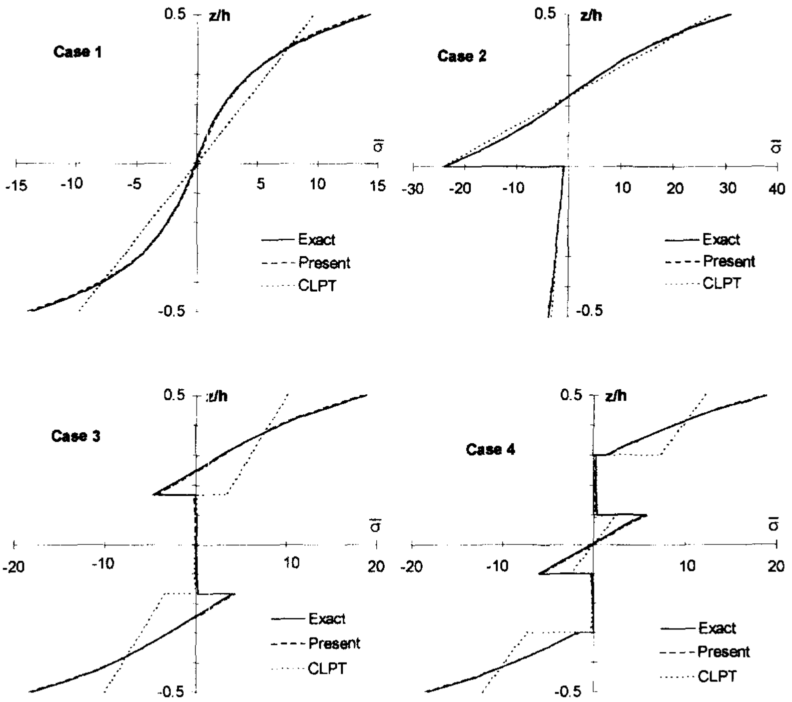

Fig. 3. Comparison of axial stresses for the various cases with $L/h = 4$ subjected to a half sine distributed loading.

solution, as shown in Figs (2a) and (2b), respectively. Since the deflection in the exact solution is not a constant value and varies over the thickness of the plate, average values were calculated and used in Fig. (2b). Comparisons of the stresses obtained by the three methods for $L/h = 4$ are also shown in Figs (3) and (4). The following normalized quantities have been defined in connection with these figures:

$$\bar{\sigma} = \frac{\sigma\left(\dfrac{L}{2}, z\right)}{q_0} \quad \bar{\tau} = \frac{\tau(0, z)}{q_0}. \tag{22}$$

As the figures show, the results of the present method are in very good agreement with the results of the exact solutions. The stresses obtained by the present method are so close to the exact values that the corresponding lines are hardly distinguishable. The margin of error for the deflections obtained by the present method in comparison with those obtained by the exact solution, except for case 2, is always less than 0.3%. In case 2 for L/h ratio less than 7 the error is comparatively higher than the other cases. Nevertheless, the maximum

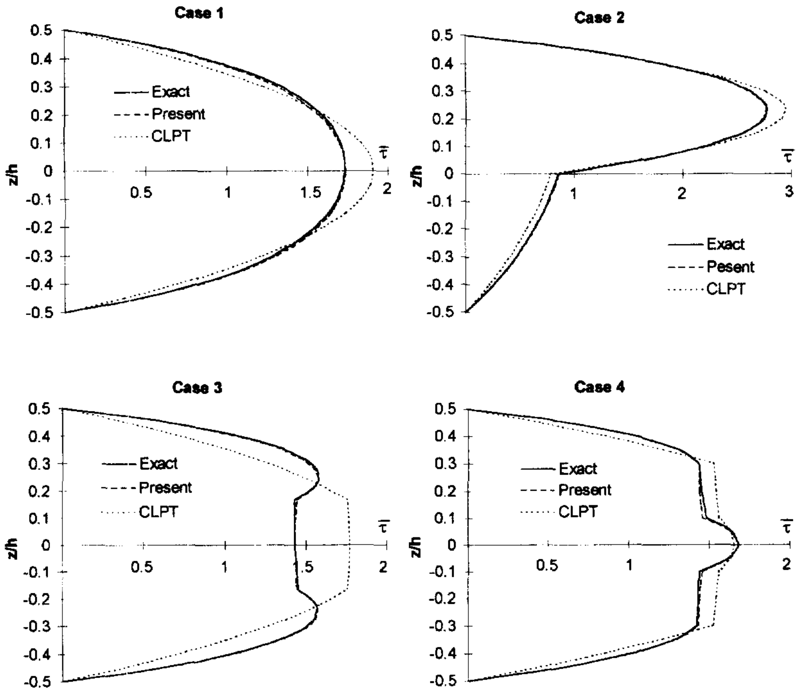

Fig. 4. Comparison of shear stresses for the various cases with $L/h = 4$ subjected to a half sine distributed loading.

error is less than 1%. It is worthwhile to note that case 2 is highly unsymmetric and it is not very common in practice.

The minute difference between the results obtained by exact solution and our proposed method is due to our fundamental assumption, which does not consider the through-the-thickness deformation of laminates. The over estimation of the deflections can be explained by the principle of energy. The energy due to applied external load is transferred into internal energy comprising of various components. For the present case study the constituents are due to $\varepsilon_x\sigma_x$, $\gamma_{xz}\tau_{xz}$ and $\varepsilon_z\sigma_z$. The first two constituents corresponds to the overall deflection of the beam and the last is the outcome of through-the-thickness deformation. Not considering the through-the-thickness deformation and, therefore, the corresponding energy, leads to an over estimation of the energy associated with the overall deflection. This, in turn, results to an over estimation of the overall deflections. Note that the results of the exact solution converge to those of the present method when the through-the-thickness stiffness of the plate increases.

The response of the laminates subjected to a concentrated load can be found by superposition of the solution of the harmonic loading given by eqn (9). The exact solution for the mathematically concentrated load, i.e. load applied on a zero width, leads to an infinite displacement at the point of application of the load. To avoid this unreal situation we assume that the load is applied through a 5 mm-diameter roller. Also as depicted in Fig. (5) the concentrated load is taken as 0.1 kN mm^{-1}. We adopt Hertzian contact law to simulate the contact behavior of the laminated plate and the roller. Therefore, the width of contact and the distribution of load over the contact width, as given by Goldsmith (1960), are:

$$2b = 4\sqrt{F(\delta_1+\delta_2)R}$$
$$q = \frac{2F}{\pi b^2}\sqrt{(b^2-\bar{x}^2)}, \qquad (23)$$

respectively, where $\bar{x}$ is measured from the center of the contact width and

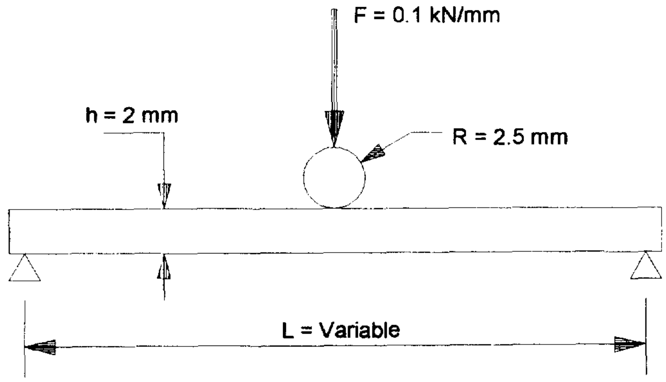

Fig. 5. Configuration of the laminates subjected to a concentrated load.

$$\delta_1 = \frac{1-\mu_1^2}{E_1\pi}$$

$$\delta_2 = \frac{1-\mu_2^2}{E_2\pi}, \tag{24}$$

where E_1, μ_1, E_2 and μ_2 are the elastic moduli and the Poisson's ratios of the two contacting bodies. Here we assume the following:

for the roller

$$E_1 = 200\ \mathrm{GPa}$$

$$\mu_1 = 0.3, \tag{25}$$

for the laminates

$$E_2 = 7\ \mathrm{GPa}$$

$$\mu_2 = 0.25. \tag{26}$$

Based on the above information, the response of the laminates of different length was analyzed. The ratio of the deflections obtained by the present method to those obtained by exact solution for midspan of the laminates are presented in Fig. (6a). In order to eliminate the local effect of the concentrated load, the deflection of the bottom line from the exact solution has been used. As the figure shows, the over-estimation of the deflections in this case is much higher than the previous case when a half a sine load was considered. The reason for such a difference and the remedy will be discussed later.

The axial stresses at the top and bottom of laminate case 1 for $L/h = 10$ have also been obtained. These stresses together with the stresses obtained from present method are shown

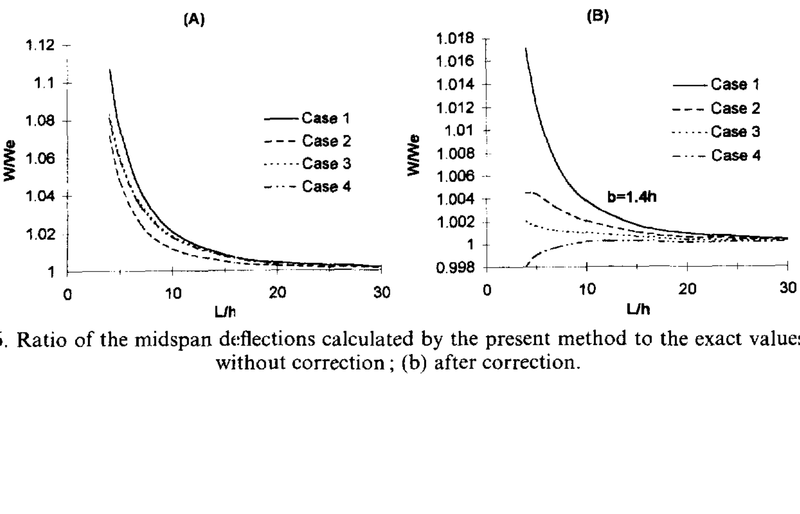

Fig. 6. Ratio of the midspan deflections calculated by the present method to the exact values: (a) without correction; (b) after correction.

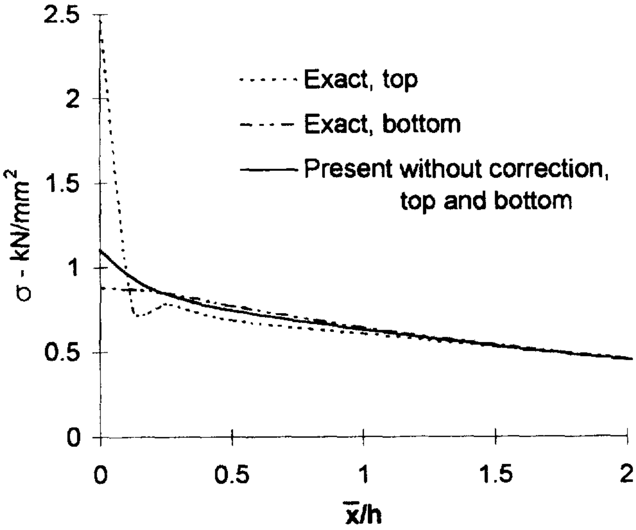

Fig. 7. Comparison of the axial stresses in the vicinity of midspan for laminate case 1 with $L/h = 10$.

in Fig. (7). In this figure, $\bar{x}$ is the distance from midspan. The local effect of the concentrated load causes a high axial stress at the top of the laminate at its midspan and some fluctuation in the vicinity of load. As $\bar{x}$ increases, the difference between the stresses obtained by the two methods vanishes. However, since the distance between the contact region at the top and the corresponding point at the bottom surface is relatively large, the difference between the values of stresses obtained by the two methods cannot be due to the local effect of the concentrated load. This problem is discussed in further detail.

Figure 8 shows the distribution of σ_z on the top and middle plane of the plate for two extreme cases (i.e. for isotropic and highly orthotropic materials) based on the exact solution. The heavily concentrated stress distribution on top of the plate changes to more distributed pattern as depth increases. The assumption of inextensibility which is the basis of the proposed method does not account for this gradual distribution of the load and, therefore, over-estimates the axial stress and the displacement at midspan. To solve this problem it is obvious that the load should be distributed over a slightly wider area. For this

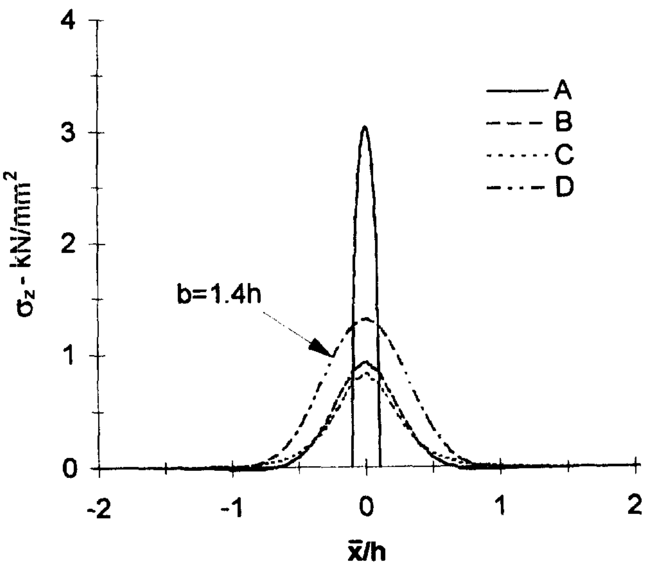

Fig. 8. Distribution of the through-the-thickness normal stress: (a) at the contact surface; (b) at midplane of an isotropic plate; (c) at midplane of a highly anisotropic plate with $E/G_{xy} = 100$; (d) based on the assumed distribution.

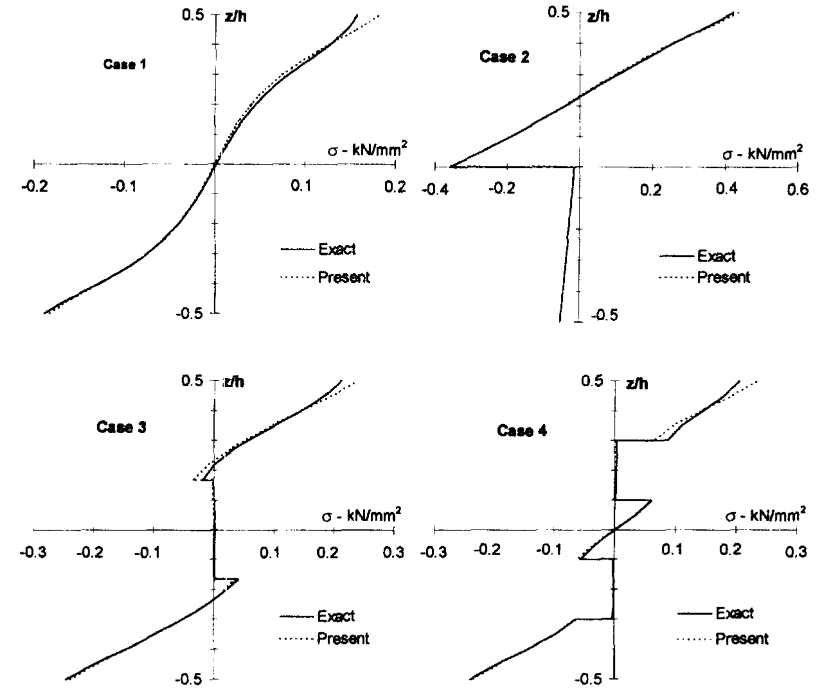

Fig. 9. Comparison of the axial stresses at a distance h from the midspan for various cases with $L/h = 4$ subjected to the concentrated load.

purpose, we assume the distribution of load to be based on the following relation. This relation is also presented by line D in Fig. (8):

$$q = \frac{F}{8.75b}[1+\cos(\pi\bar{x}/b)]^4, \tag{27}$$

where b is half of the assumed width for distribution of the load. The value of b depends on the degree of anisotropy of the material of the plate and increases as the anisotropy increase. For most practical purposes it lies between $1.3h$ and $1.6h$. However, the result is not very sensitive when the value of b is taken within these limits. Here we assume a value of $b = 1.4h$.

Based on this new distribution of the load, given by eqn (27), the response of the laminates is calculated accordingly. The ratio of the displacements obtained in this case to the exact values are shown in Fig. 6. The discrepancies in this case are quite small. Since case 1 is highly anisotropic a better result can be obtained by taking $b = 1.6h$. The comparison of the stresses obtained by the two methods, at a section h apart from midspan is also shown in Figs 9 and 10. By increasing the distance from the midspan, the small discrepancy between the results of the two methods vanishes.

To include the local effect of the concentrated load in the vicinity of the midspan, we consider the square block shown in Fig. 11. The shear and the axial stresses of the upper half of this block will be added to the results obtained previously. The solution of this problem even for isotropic material (Goodier, 1932) is mathematically challenging, therefore, an approximate solution is proposed. For this purpose we use the solution of an anisotropic strip subjected to a harmonic loading (see Fig. 12). The solution is as follows:

$$\tau = \frac{q_0}{p}\cos px \frac{\dfrac{\sinh(m_1 y)}{\sinh(m_1 h)} - \dfrac{\sinh(m_2 y)}{\sinh(m_2 h)}}{\dfrac{\cosh(m_2 h)}{m_1 \sinh(m_1 h)} - \dfrac{\cosh(m_2 h)}{m_2 \sinh(m_2 h)}} \tag{28}$$

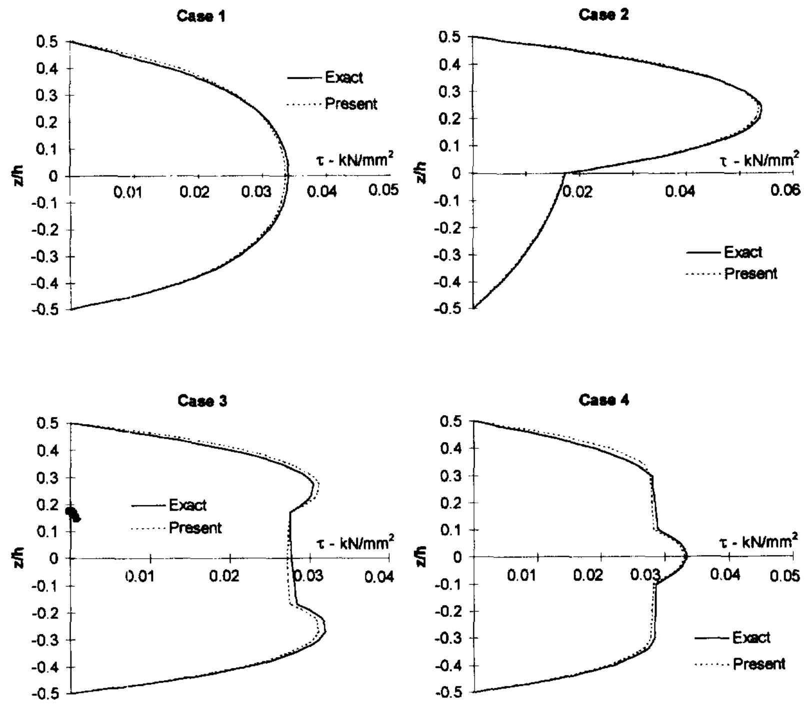

Fig. 10. Comparison of the shear stresses at a distance h from midspan for various cases with $L/h = 4$ subjected to the concentrated load.

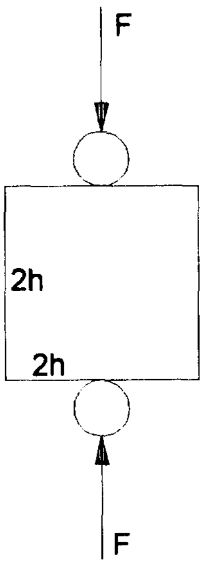

Fig. 11. Configuration of the block representing the local effect.

$$\sigma = -\frac{q_0}{p^2}\sin px\frac{\dfrac{m_1\cosh(m_1y)}{\sinh(m_1h)}-\dfrac{m_2\cosh(m_2y)}{\sinh(m_2h)}}{\dfrac{\cosh(m_1h)}{m_1\sinh(m_1h)}-\dfrac{\cosh(m_2h)}{m_2\sinh(m_2h)}} \tag{29}$$

where

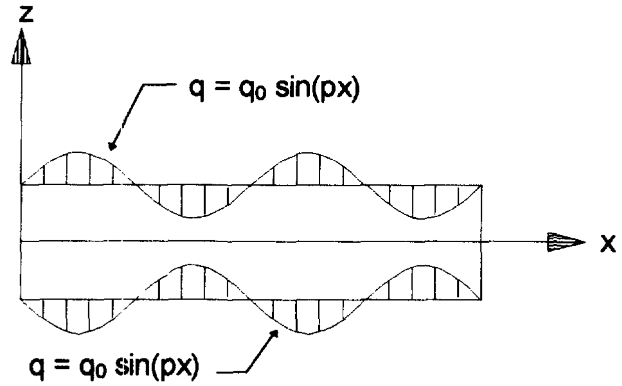

Fig. 12. Anisotropic strip under harmonic loading.

$$m_1 = p\sqrt{\frac{\bar{a}+\bar{b}}{\bar{c}}}$$

$$m_2 = p\sqrt{\frac{\bar{a}-\bar{b}}{\bar{c}}}, \tag{30}$$

and

$$\bar{a} = R_{66}+2R_{13}$$
$$\bar{b} = \sqrt{\bar{a}^2-4R_{11}R_{33}}$$
$$\bar{c} = 2R_{11}. \tag{31}$$

R_{ij} are the reduced compliance coefficients for plane strain, defined by:

$$R_{ij} = S_{ij}-\frac{S_{i2}S_{j2}}{S_{22}} \quad \text{for} \quad i,j = 1,3,6, \tag{32}$$

where S_{ij} are the compliances with respect to the axes of material symmetry.

The problem encountered with this solution is that at the two boundaries, $x = 0$ and $x = 2h$, the shear stresses are not zero. To resolve this problem, we apply the following approximate correction:

$$\tau^* = \tau-\frac{h-x}{h}\tau(x = 0), \tag{33}$$

where τ is the value obtained from eqn (28).

The solution for the block subjected to a concentrated load is obtained by the superposition of the above solution for harmonic loading. The property of the upper most layer of the laminate is taken for this purpose. The results of the correction on the stresses of the sections in vicinity of concentrated load are shown in Figs 13–15. As the figures show, the accuracy of the results for sections close to midspan and for 0° layers is very good. The local effect of the concentrated load is more pronounced at midspan for axial stresses and at sections passing approximately at the end of the contact area for shear stresses. These are also the sections which carry the critical stresses. As the figures show the critical stress values from two methods compare very well with each other. The discrepancy of the results for uncritical values is considerable. Nevertheless, since in practice these values are not the controlling parameters, the discrepancy of the results on these values has minor importance.

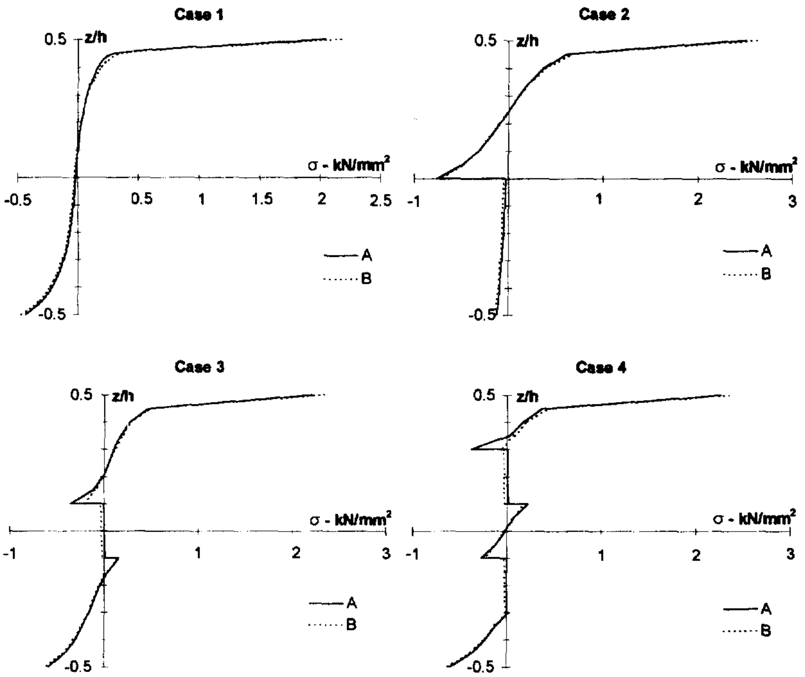

Fig. 13. Comparison of the axial stresses at midspan for the various cases with $L/h = 4$ subjected to concentrated load: (a) exact values; (b) present method including the local effect of concentrated load.

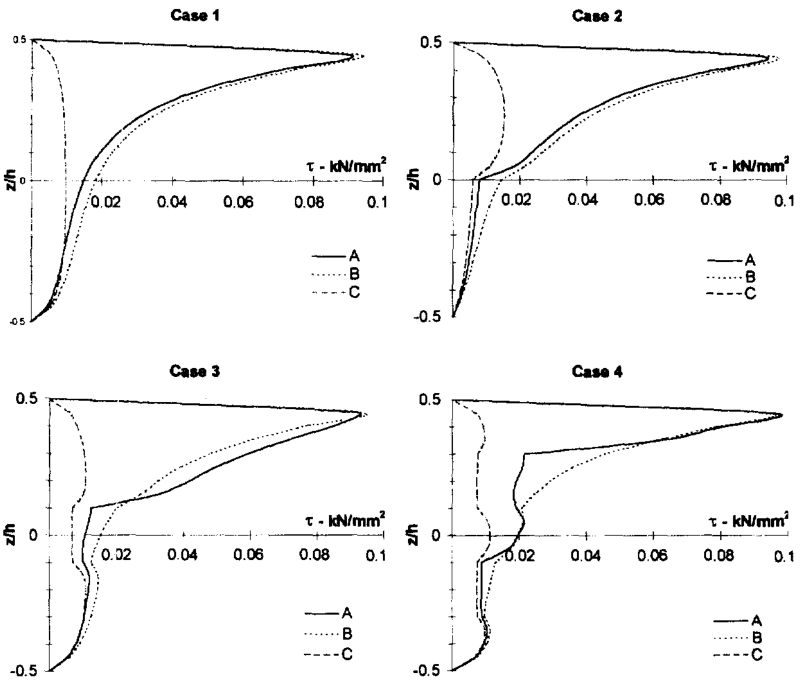

Fig. 14. Comparison of the shear stresses at $0.125h$ from midspan for the various cases with $L/h = 4$ subjected to concentrated load: (a) exact solution; (b) present method including the local effect of the concentrated load; (c) present method without the local effect.

SUMMARY AND CONCLUSION

The response of cross-ply laminated plates under cylindrical and/or planar bending was examined. A general form of an in-plane displacement field has been obtained by the assumption of the through-the-thickness inextensibility and the integration of engineering shear strains in the z-direction. The behavior of the displacement field depends on the

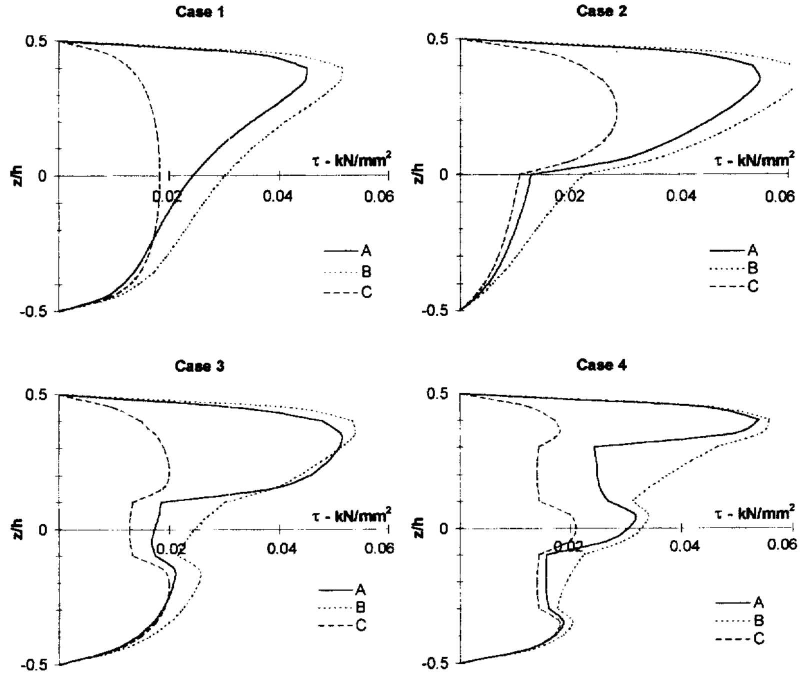

Fig. 15. Comparison of the shear stresses at $0.25h$ from the midspan for the various cases with $L/h = 4$ subjected to concentrated load: (a) exact solution; (b) present method including the local effect of the concentrated load; (c) present method without the local effect.

distribution of shear stress through the thickness of the plate. The displacement field used in the classical plate theory and some of the higher-order theory can be obtained from the proposed displacement field by appropriate assumption of shear distribution. In the present work, the exact distribution of shear stress, and therefore, the complete solution has been derived for a laminated plate simply supported at its two ends. The proposed solution was validated by analyzing plates with four different layup sequences as follows: a one-layer 0° unidirectional, a two-layer, a three-layer and a five-layer cross-ply laminates. Each plate was examined under two loading schemes: (a) a half sine distributed load, and (b) a concentrated load. A simple approximate procedure was used to account for the effect of the local stresses resulting from the application of the concentrated load. The behavior of the solution can be summarized as follows:

(1) The predicted deflection of the plates under a half sine loading are slightly higher than the exact values (Pagano, 1969). The difference for all cases, with the exception of the second layup sequence with $L/h < 7$, is less than 0.3%. For laminate case 2, which is highly unsymmetric, the predicted deflection is about 0.9% higher for $L/h = 4$. These minute discrepancies vanish as the ratio of L/h increases.

(2) The predicted values of the stresses for the half a sine loading are practically identical to those calculated by the exact solution.

(3) The assumption of the through-the-thickness inextensibility does not account for the gradual distribution of the concentrated load over the thickness of the plate and, therefore, the distribution of load based on the contact law produces erroneously larger values for tensile stress and deflections in the vicinity of the load region. However, one can obtain excellent improvements in the predicted results when distributing the load over a slightly larger area.

(4) The predicted stresses for sections at a distance equal to the thickness of the laminate from the position of concentrated load are in good agreement with the exact values. The minute discrepancy vanishes as the distance increases.

(5) The predicted stresses in the vicinity of concentrated load for 0° layers have high accuracy. However, those of the 90° layers are not satisfactory. Recognizing that the critical stresses estimated by the present method in the vicinity of the concentrated load are in good agreement with the exact values, practically, the other discrepancies are not significant.

(6) For a laminate consisting of n layers, with the present method one needs to solve n simultaneous equations, which is four times less than that required for the exact solution.

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